OPTIMUM STRUCTURAL DESIGN BASED ON EXTENDED RELIABILITY THEORY

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Abstract

This paper deals with an optimum design of structural systems taking account of various statistical variations in strengths of materials, applied loads, fabrication processes, *etc.*, and subjective uncertainties associated with engineering judgements. Second-moment approximation is applied to reliability analysis of structural systems and an optimum design problem is set up to determine an optimum structure minimizing the structural weight based on *safety index formats*. The problem is effectively solved by using a nonlinear programming technique called SLP (Sequential Linear Programming). Numerical examples are presented to demonstrate the validity of the proposed design procedure.

1. Introduction

Structural designs are to be made from the statistical point of view[1] since many uncertainties[2] are involved in the designs, and many studies have been made of the optimum designs[3~14] However, these approaches have not been fully implemented in design practice, partly because there are no efficient methods for calculating multidimensional probability distribution functions[10,14~16] and partly because information on the statistical properties of resistances and loads is limited. Recently, efforts aimed at the implementation of reliability-based design have been made, and reliability analysis and design using the second moment approximation[2,17,18] is being accepted as a feasible method. Some tentative designs have been made of basic structural elements.[19,20] However, these concepts are not fully extended to the structural systems.[12,21]

This paper deals with an optimum design of structural systems when various uncertainties encountered in structural designs, such as those of allowable stresses of materials, applied loads, dimensions of structural elements due to fabrication errors, mathematical models in stress analysis, *etc.*, are taken into account. By using first order approximation, safety index formats are presented for reliability evaluation of structural designs. The optimum design problem is set up to determine the structural system minimizing the structural weight based on the safety index formats. The problem is effectively solved by employing a nonlinear programming technique called SLP (Sequential Linear Programming). Numerical examples are provided to demonstrate the validity of the proposed method.

2. Mathematical Model of Structural System

Consider a structural system which consists of n elements with specified configuration. It is assumed that there exist m failure modes and the safety margins (Z_i) are expressed in the form :

$$\begin{aligned} Z_i &= T_i(R_1,R_2,\ldots,R_n)-S_i(L_1,L_2,\ldots,L_l) \\ \text{or } &= C_{yi} - U_i(X_1,X_2,\ldots,X_n,L_1,L_2,\ldots,L_l) \\ & \qquad (i=1,2,\ldots,m) \end{aligned} \tag{1}$$

where T_i = resultant strength of the i-th failure mode
S_i = resultant load of the i-th failure mode
C_{yi} = allowable stress of the i-th element
U_i = applied stress of the i-th element
R_j = strength of the j-th element
X_j = dimension such as cross-sectional area, thickness, *etc.*
L_j = load acting on the structure

Failure of the structural system is assumed to occur if any one of the safety margins is negative, *i.e.*, $Z_i<0$. The strengths of the structural elements are determined by the allowable stress of the materials to be used and their dimensions, such as cross-sectional areas, thickness, *etc.*, and they are given by

$$R_j = R_j(C_{yj},X_j) \qquad (j=1,2,\ldots,n) \tag{2}$$

There are many factors of variability in a structural design. It is a wellknown fact that the strengths of the materials have statistical variations due to variability in purity and composition of their constituents and in manufacturing processes. The loads are not always applied to the structure as predicted in the design stage. The dimensions may also deviate from the specified values because of fabrication errors.

These factors are modeled as random variables with appropriate distribution. However, data for describing their distributions are not so amply provided that their distributions can be exactly specified. In practice, information may be limited to their first- and second-order moments. Consequently, correcting factors ($N_{(\cdot)}$)[17,18] are introduced to compensate the errors in modeling the random variables:

$$\begin{aligned} C_{yj} &= N_{C_{yj}} \hat{C}_{yj} \\ L_j &= N_{L_j} \hat{L}_j \\ X_j &= N_{X_j} \hat{X}_j \end{aligned} \tag{3}$$

where $\hat{(\cdot)}$ denotes the theoretical model of $(\cdot)$.

Similarly, the functional forms of the safety margins and the strengths of the structural elements are expressed, taking account of imperfections in modeling:

$$
\begin{aligned}
T_i &= N_{Ti}\hat{T}_i \\
S_i &= N_{Si}\hat{S}_i \\
U_i &= N_{Ui}\hat{U}_i \\
R_j &= N_{Rj}\hat{R}_j
\end{aligned}
\tag{4}
$$

The correcting factors given above permit also to take account of modeling errors due to engineering judgements in stress analysis of the structural system.

When the configuration of the structural system and the materials to be used are specified, the structural weight is given by

$$W = W(X_1, X_2, \ldots, X_n) \tag{5}$$

3. Reliability Evaluation by Safety Index

As mentioned in the preceding section, data for describing the random variables and the functional forms of the quantities concerning the safety margins are not so complete that they are correctly predicted. The statistical measures, which are consistent with the information available in practice and convenient for application, are the means and variances or alternatively coefficients of variation. Consequently, the following assumptions are made of the various uncertainties:

(a) The means and coefficients of variation are given of random variables, *i.e.*, strengths of materials, dimensions and loads.
(b) The correcting factors $N_{(\cdot)}$ have means 1.0 and specified values of coefficients of variation.

Using first order approximation, the mean and standard deviation of the safety margin are calculated as follows:

$$
\begin{aligned}
\bar{Z}_i &= \bar{T}_i - \bar{S}_i \quad \text{or} \quad \bar{C}_{yi} - \bar{U}_i \\
\sigma^2_{Zi} &= \sigma^2_{Ti} + \sigma^2_{Si} \quad \text{or} \quad \sigma^2_{Cyi} + \sigma^2_{Ui}
\end{aligned}
\tag{6}
$$

where

$$\bar{T}_i = \hat{T}_i(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n)$$

$$\bar{S}_i = \hat{S}_i(\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_l)$$

$$\bar{U}_i = \hat{U}_i(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n, \bar{L}_1, \bar{L}_2, \ldots, \bar{L}_l)$$

$$\sigma^2_{Ti} = (\Delta N_{Ti})^2(\bar{T}_i)^2 + \sum_{j=1}^{n}(\hat{T}_{ij})^2\sigma^2_{Rj}$$

$$\sigma^2_{Si} = (\Delta N_{Si})^2(\bar{S}_i)^2 + \sum_{j=1}^{l}(\hat{S}_{ij})^2\sigma^2_{Si}$$

$$\sigma^2_{Ui} = (\Delta N_{Ui})^2(\bar{U}_i)^2 + \sum_{j=1}^{n}(\hat{U}_{i\bar{j}})^2\sigma^2_{Xj} + \sum_{j=1}^{l}(\hat{U}_{i\bar{j}})^2\sigma^2_{Lj}$$

$$\sigma^2_{Rj} = (\Delta N_{Rj})^2(\bar{R}_j)^2 + \sum_{j=1}^{n}[(\hat{R}_{i\bar{j}})^2\sigma^2_{Cyj} + (\hat{R}_{i\bar{j}})^2\sigma^2_{Xj}]$$

$$\sigma^2_{Cyj} = (\Delta C_{yj})^2(\bar{C}_{yj})^2; \qquad (\Delta C_{yj})^2 \triangleq (\Delta N_{Cyj})^2 + (\Delta\hat{C}_{yj})^2$$

$$\sigma^2_{Xj} = (\Delta X_j)^2(\bar{X}_j)^2; \qquad (\Delta X_j)^2 \triangleq (\Delta N_{Xj})^2 + (\Delta\hat{X}_j)^2$$

$$\sigma^2_{Lj} = (\Delta L_j)^2(\bar{L}_j)^2; \qquad (\Delta L_j)^2 \triangleq (\Delta N_{Lj})^2 + (\Delta\hat{L}_j)^2$$

$\Delta(\cdot) = \sigma_{(\cdot)}/(\bar{\cdot})$ = coefficient of variation in $(\cdot)$

$\sigma_{(\cdot)}$ = standard deviation of $(\cdot)$

$(\bar{\cdot})$ = mean of $(\cdot)$

$\hat{T}_{ij} = \dfrac{\partial\hat{T}_i}{\partial R_j}$ evaluated at $R_j = \bar{R}_j$

$\hat{S}_{ij} = \dfrac{\partial\hat{S}_i}{\partial L_j}$ evaluated at $L_j = \bar{L}_j$

$\hat{U}_{i\bar{j}} = \dfrac{\partial\hat{U}_i}{\partial X_j}$ evaluated at $X_j = \bar{X}_j$ and $L_j = \bar{L}_j$

$\hat{U}_{i\bar{j}} = \dfrac{\partial\hat{U}_i}{\partial L_j}$ evaluated at $X_j = \bar{X}_j$ and $L_j = \bar{L}_j$

$\hat{R}_{i\bar{j}} = \dfrac{\partial\hat{R}_i}{\partial C_{yj}}$ evaluated at $C_{yj} = \bar{C}_{yj}$ and $X_j = \bar{X}_j$

$\hat{R}_{i\bar{j}} = \dfrac{\partial\hat{R}_i}{\partial X_j}$ evaluated at $C_{yj} = \bar{C}_{yj}$ and $X_j = \bar{X}_j$

It is seen from the above relations that the correcting factors of the random variables (N_{Cyj}, N_{Xj}, N_{Lj}) contribute only to increase in their resultant coefficients of variation. Hence, they are treated in the following by embedding them in variability in coefficients of variation of the random variables.

Safety index formats require that the mean of the safety margin ($\bar{Z}_i$) must be larger than λ_i times standard deviation (σ_{Zi}), *i.e.*,

$$\bar{Z}_i \geq \lambda_i \sigma_{Zi} \qquad (i = 1, 2, \ldots, m) \tag{7}$$

Eq.(7) enables us to take account of the various uncertainties in a structural design by considering convenience of design and availability of statistical information.

The central factor of safety of the i-th failure mode, *i.e.*,

$$SF_i \triangleq \bar{T}_i/\bar{S}_i \quad \text{or} \quad \bar{C}_{yi}/\bar{U}_i \tag{8}$$

is given by

$$\{1 + \lambda_i\sqrt{(\Delta T_i)^2 + (\Delta S_i)^2 - (\lambda_i \Delta T_i \Delta S_i)^2}\}/\{1 - (\lambda_i \Delta T_i)^2\}$$

or

$$\{1 + \lambda_i\sqrt{(\Delta C_{yi})^2 + (\Delta U_i)^2 - (\lambda_i \Delta C_{yi} \Delta U_i)^2}\}/\{1 - (\lambda_i \Delta C_{yi})^2\}$$

when equality holds in Eq.(7).

Eq.(7) is shown in Appendix 1 to be equivalent to the probability constraint on the failure probability of the i-th failure mode:

$$\text{Prob}[Z_i \leq 0] \leq P_{fai} \tag{9}$$

where P_{fai} is the specified allowable failure probability. Hence, the safety index λ_i is related to the allowable failure probability (P_{fai}) when the probability distribution function of Z_i is known.

4. Optimum Design of Structural Systems

For the optimum design of structural systems, the following assumptions are further made:

(1) The configuration of the systems is predetermined. That is, the length of the structural elements and load conditions are specified, and thus the variables left to be determined are cross-sectional areas, thickness, *etc.*

(2) The materials to be used are specified and the means and coefficients of variation in their allowable stresses are given.

(3) The variables to be determined, such as cross-sectional areas, thickness, *etc.*, are random variables with specified values of coefficients of variation. Consequently, their mean values ($\bar{X}_j$) are taken as design variables.

(4) The weight of the structural system is evaluated by the design variables, *i.e.*,

$$W = W(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n) \triangleq W(\bar{X}) \tag{10}$$

where $\bar{X}$ denotes an n-dimensional vector $(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n)^T$.

The optimum design problem is stated as follows:

PROBLEM: Determine the optimum values of the design variables to minimize the structural weight Eq.(10) for the specified values of safety indexes to the safety margins, *i.e.*,

Minimize: $W(\bar{X})$

with respect to $\bar{X}$ under the constraints:

$$\bar{Z}_i(\bar{X}) \geq \lambda_i \sigma_{Z_i}(\bar{X}) \quad (i=1,2,\ldots,m) \tag{11}$$

where λ_i are the given constants and the expressions $\bar{Z}_i(\bar{X})$ and $\sigma_{Z_i}(\bar{X})$ are used to indicate that they are functions of $\bar{X}$.

This is a nonlinear programming problem and also interpreted as a deterministic equivalent of a stochastic programming problem since the probability constraints Eq.(9) are reduced to the constraints Eq.(11) as shown in Appendix 1.

The problems have been solved by using SUMT(Sequential Unconstrained Minimization Technique)[22] combined with Fletcher-Reeves method and SLP(Sequential Linear Programming)[22] which is briefly illustrated in Appendix 2. The former is found to be ineffective due to enormous computation time required for the problems with design variables more than four while SLP very effective even for large scale problems. Hence, SLP is used in the following examples.

5. Numerical Examples

Numerical examples are presented to illustrate the design procedures.

Example 1. Box beam. Consider a simple box beam whose cross section is shown in Fig.1. The stringers and webs are designated as i and $\underline{i}$ (i= 1,2,3, 4). The bending stresses of the stringers are given by[23]

$$\hat{U}_i = \frac{M_x I_{xz} - M_z I_x}{I_x I_z - I_{xz}^2} x_i + \frac{M_z I_{xz} - M_x I_z}{I_x I_z - I_{xz}^2} z_i \quad (i=1,2,3,4)$$

where I_x and I_z are moments of inertia of the cross-sectional area, I_{xz} product of inertia, x_i and z_i coordinates of the i-th stringer with respect to the centroidal axes and M_x and M_z external bending moments.

The shear stresses in the webs are given by

$$\hat{U}_{i+4} = (q_0 + q_i)/t_i \quad (i=1,2,3,4)$$

where $q_0 = (T - 2\sum_{i=1}^{4} B_i q_i)/2B$

$$q_i = \sum_{j=1}^{i} \left(\frac{S_z I_{xz} - S_x I_x}{I_x I_z - I_{xz}^2} x_j + \frac{S_x I_{xz} - S_z I_z}{I_x I_z - I_{xz}^2} z_j\right) A_j$$

A_i = cross-sectional area of the i-th stringer

t_i = thickness of the i-th web

B_i = area enclosed by the i-th web and the lines to the centroid from the adjacent stringers

$B = \sum_{i=1}^{4} B_i$

S_x, S_z = external shearing forces

T = external torsional moment

The safety margins are given by

$$Z_i = C_{yi} - |U_i|, \quad Z_{i+4} = C_{ysi} - |U_{i+4}| \quad (i=1,2,3,4)$$

where C_{yi} and C_{ysi} are the allowable stresses of the i-th stringer and web.

The weight per unit length is given by

$$W = \sum_{i=1}^{4} (\rho_i A_i + \rho_{si} l_i t_i)$$

where l_i are the lengths of the webs and $\rho_{(\cdot)}$ the densities.

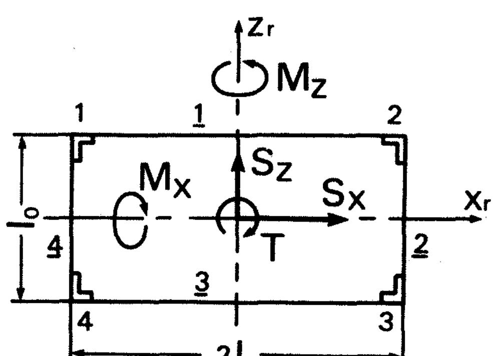

Fig. 1 Box beam (X_r, Z_r: reference axes)

The means of the cross-sectional areas of the stringers and the thickness of the webs are taken as the design variables. The data concerned are listed in Table 1.

First, consider a case where no modeling errors are considered. The optimum solutions are listed in Table 2 for various values of safety indexes. The central factors of the failure modes defined by Eq.(8) are also given in the table. It is seen that the optimum solutions move to the safety side and thus the structural weight becomes heavy as the values of the safety indexes are increased.

Table 3 shows the optimum designs in the case where the modeling errors are taken account of. By comparing the results with those of Table 2 in the same values of the safety indexes, the optimum values of the design variables are large when the modeling errors are considered. Consequently, the designs with modeling errors neglected result in insufficient structural reliability.

Table 4 indicates the optimum solutions corresponding to λ_i=4.0 when the coefficients of variation in allowable stresses, dimensions, loads and modeling errors are changed from reference conditions of Table 1. It is seen that the optimum designs are sensitive to variability in the coefficients of variation. The resulting designs are also influenced by the load conditions as illustrated in Table 5.

Table 1 Data of box beam

(1) Data of materials

Member	Material	Allowable stress C_{y_j} ×10^8 *Pa*	ΔC_{y_j}	Density ρ_i x10^3 *kg/m*^{3}
Stringers 1,2,3,4	7075-T6	4.51	0.05	2.80
Upper web 1	7075-T6	3.04	0.05	2.80
Lower web 3	2024-T3	2.75	0.05	2.76
Side web 2,4	2024-T3	2.75	0.05	2.76

(2) Data of loads

	M_x *kN·m*	M_z *kN·m*	T *kN·m*	S_x *kN*	S_z *kN*
Mean	68.6	15.7	4.41	19.6	49.0
C.O.V*	0.1	0.1	0.1	0.1	0.1

* Coefficient of variation

(3) Coefficients of variation in design variables : ΔA_j=0.02, Δt_j=0.02 (j=1,2,3,4)

(4) Distance between stringers : l_0=240 *mm*

Table 2 Optimum solutions for various values of safety indexes when no modelling errors are considered (Box beam, ΔN_{Ui}=0.0)

λ_i	$\bar{A}_1$ *mm*^{2}	$\bar{A}_2$ *mm*^{2}	$\bar{A}_3$ *mm*^{2}	$\bar{A}_4$ *mm*^{2}	$\bar{t}_1$ *mm*	$\bar{t}_2$ *mm*	$\bar{t}_3$ *mm*	$\bar{t}_4$ *mm*	W *kg/m*
0	284 (1.00)	350 (1.00)	278 (1.00)	356 (1.00)	0.13 (1.00)	0.30 (1.00)	0.01 (1.00)	0.44 (1.00)	4.193
1	350 (1.13)	358 (1.11)	284 (1.13)	424 (1.11)	0.13 (1.09)	0.32 (1.13)	0.03 (1.43)	0.51 (1.11)	4.666
2	398 (1.26)	388 (1.22)	311 (1.26)	475 (1.22)	0.13 (1.19)	0.36 (1.27)	0.04 (1.75)	0.56 (1.22)	5.182
3	440 (1.44)	428 (1.34)	349 (1.41)	519 (1.34)	0.15 (1.30)	0.40 (1.42)	0.05 (2.20)	0.62 (1.34)	5.731
4	492 (1.55)	465 (1.47)	382 (1.56)	574 (1.47)	0.16 (1.42)	0.44 (1.57)	0.06 (2.49)	0.68 (1.47)	6.319
5	527 (1.72)	525 (1.61)	439 (1.72)	613 (1.61)	0.18 (1.54)	0.50 (1.75)	0.07 (3.51)	0.73 (1.61)	6.955

Note: Numbers in brackets designate central factors of safety.

Table 3 Optimum solutions for various values of safety indexes when modelling errors are considered (Box beam, ΔN_{Ui}=0.1)

λ_i	$\bar{A}_1$ *mm*^{2}	$\bar{A}_2$ *mm*^{2}	$\bar{A}_3$ *mm*^{2}	$\bar{A}_4$ *mm*^{2}	$\bar{t}_1$ *mm*	$\bar{t}_2$ *mm*	$\bar{t}_3$ *mm*	$\bar{t}_4$ *mm*	W *kg/m*
1	361 (1.16)	371 (1.15)	292 (1.16)	440 (1.15)	0.13 (1.14)	0.33 (1.17)	0.03 (1.44)	0.53 (1.15)	4.824
2	424 (1.33)	410 (1.30)	325 (1.34)	509 (1.30)	0.14 (1.28)	0.38 (1.34)	0.05 (1.73)	0.60 (1.30)	5.500
3	455 (1.51)	486 (1.46)	393 (1.51)	548 (1.46)	0.17 (1.43)	0.44 (1.53)	0.05 (2.79)	0.60 (1.46)	6.213
4	535 (1.70)	520 (1.63)	419 (1.70)	635 (1.64)	0.18 (1.60)	0.48 (1.71)	0.06 (2.61)	0.66 (1.64)	6.963
5	610 (1.90)	566 (1.81)	457 (1.91)	718 (1.82)	0.19 (1.77)	0.53 (1.91)	0.08 (2.78)	0.85 (1.82)	7.762

Note: Numbers in brackets designate central factors of safety.

Table 4 Effect of variability on optimum solutions (Box beam , λ_i=4.0)

$\Delta C_{yi}, \Delta C_{ysi}$	ΔL_j	ΔX_j	ΔN_{Ui}	$\bar{A}_1$ mm^2	$\bar{A}_2$ mm^2	$\bar{A}_3$ mm^2	$\bar{A}_4$ mm^2	$\bar{t}_1$ mm	$\bar{t}_2$ mm	$\bar{t}_3$ mm	$\bar{t}_4$ mm	W kg/m
0.05	0.1	0.02	0.1	535 (1.70)	520 (1.63)	419 (1.70)	635 (1.64)	0.18 (1.60)	0.48 (1.71)	0.06 (2.61)	0.75 (1.64)	6.963
0.1	0.1	0.02	0.1	656 (2.01)	597 (1.95)	472 (2.01)	781 (1.95)	0.20 (1.92)	0.56 (2.02)	0.08 (2.55)	0.92 (1.96)	8.268
0.05	0.2	0.02	0.1	647 (2.10)	615 (1.91)	523 (2.10)	737 (1.92)	0.20 (1.79)	0.60 (2.12)	0.10 (4.35)	0.88 (1.91)	8.340
0.05	0.1	0.04	0.1	535 (1.71)	526 (1.64)	425 (1.71)	636 (1.65)	0.18 (1.62)	0.49 (1.73)	0.07 (2.78)	0.76 (1.65)	7.014
0.05	0.1	0.02	0.2	617 (2.00)	641 (1.96)	510 (2.01)	747 (1.97)	0.23 (1.94)	0.58 (2.02)	0.06 (3.12)	0.90 (1.96)	8.299

Note: Numbers in brackets designate central factors of safety.

Table 5 Effect of load conditions on optimum solutions (Box beam , λ_i=4.0)

$\bar{M}_x$ $kN\cdot m$	$\bar{M}_z$ $kN\cdot m$	$\bar{T}$ $kN\cdot m$	$\bar{S}_x$ kN	$\bar{S}_z$ kN	$\bar{A}_1$ mm^2	$\bar{A}_2$ mm^2	$\bar{A}_3$ mm^2	$\bar{A}_4$ mm^2	$\bar{t}_1$ mm	$\bar{t}_2$ mm	$\bar{t}_3$ mm	$\bar{t}_4$ mm	W kg/m
68.6	15.7	4.41	19.6	49.0	535 (1.70)	520 (1.63)	419 (1.70)	635 (1.64)	0.18 (1.60)	0.48 (1.71)	0.06 (2.61)	0.75 (1.64)	6.963
68.6	15.7	7.85	29.4	68.6	603 (1.70)	454 (1.63)	354 (1.71)	700 (1.64)	0.25 (1.60)	0.58 (1.72)	0.11 (2.24)	1.15 (1.64)	7.451
98.1	23.6	4.41	19.6	49.0	756 (1.70)	750 (1.63)	599 (1.70)	907 (1.64)	0.18 (1.60)	0.49 (1.71)	0.06 (2.68)	0.75 (1.63)	9.455
98.1	23.6	7.85	29.4	68.6	789 (1.70)	717 (1.63)	566 (1.71)	939 (1.64)	0.28 (1.60)	0.62 (1.73)	0.09 (3.39)	1.11 (1.63)	9.952

Note: (1) All the values of coefficients of variation in loads are 0.1 while other data are kept to be the same as in Table 1.
(2) Numbers in brackets designate central factors of safety.

Example 2. Engine bed. Fig.2 represents an engine bed for V-type engines, which consists of the tubular members. The loads in all the members of the structure are

$$\hat{S}_1 = -4L_1-(2/3)L_2, \quad \hat{S}_2 = L_1-(2/3)L_2$$

$$\hat{S}_3 = 2\sqrt{3}L_1+(\sqrt{3}/3)L_2, \quad \hat{S}_4 = -L_1+(2/3)L_2$$

$$\hat{S}_5 = -3L_1-(4/3)L_2$$

The strength of the member in tension is given by

$$\hat{R}_i = C_{yti}(2\pi t_i r_i)$$

where C_{yti}, t_i and r_i are the yield stress, thickness and radius of the i-th member, respectively. Compressive instability is considered for the members in compression and the compressive strength for a cylinder of radius r_i with thickness t_i is given by

$$\hat{R}_i = C_{yci}E_i\left(\frac{t_i}{r_i}\right)(2\pi t_i r_i)$$

where C_{yci} and E_i are the stability constant and Young's modulus of the i-th member.

The weight is given by

$$W = \sum_{i=1}^{5} \rho_i l_i (2\pi t_i r_i)$$

where l_i are the lengths of the members.

The design problem is to determine the optimum value of thickness of the cylindrical members when the radii are specified. The data concerned are given in Table 6.

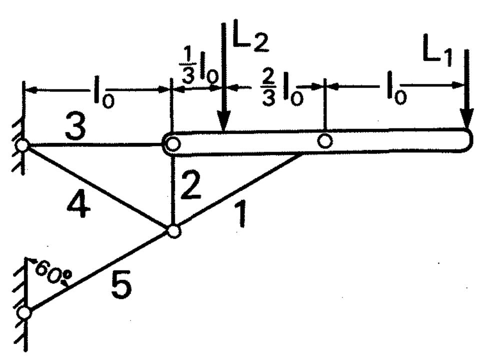

Fig. 2 Engine bed

The optimum solutions are shown in Table 7 for various values of safety indexes. It is seen that the optimum values of the design variables become large as the safety indexes are increased. Fig.3 shows the weight increases when the coefficients of variation in allowable stresses, stability constants and loads are increased from the reference conditions given in Table 6. The optimum designs are known to be influenced by the values of coefficients of variation in those random variables.

Table 6 Data of engine bed

(1) Data of 6061-T6

Proof stress $\bar{C}_{yti}\times10^8 Pa$	ΔC_{yti}	Young's modulus $E_i\times10^{10}Pa$	Density $\rho_i\times10^3 kg/m^3$
2.76	0.05	6.86	2.70

(2) Radius of members

r_1 mm	r_2 mm	r_3 mm	r_4 mm	r_5 mm
30.0	10.0	45.0	15.0	30.0

(3) Stability constants of compression members : $\bar{C}_{yci}$=0.4, ΔC_{yci}=0.1 (i=1,4,5)

(4) Length of members : l_0=750 mm

(5) Data of loads
$\bar{L}_1$=1350 N, $\bar{L}_2$=1800 N, ΔL_j=0.1 (j=1,2)

(6) Coefficients of variation in modelling : ΔN_{Ti}=0.02, ΔN_{Si}=0.02 (i=1,2,...,5)

(7) Coefficients of variation in thickness :
ΔN_{ti}=0.02 (i=1,2,...,5)

Table 7 Optimum solutions for various values of safety indexes (Engine bed)

λ_i	$\bar{t}_1$ mm	$\bar{t}_2$ mm	$\bar{t}_3$ mm	$\bar{t}_4$ mm	$\bar{t}_5$ mm	W kg
0	0.61 (1.00)	0.08 (1.00)	0.72 (1.00)	0.09 (1.00)	0.61 (1.00)	0.997
1	0.66 (1.15)	0.19 (2.22)	0.80 (1.11)	0.14 (2.23)	0.65 (1.14)	1.100
2	0.71 (1.33)	0.29 (3.46)	0.88 (1.23)	0.17 (3.53)	0.69 (1.32)	1.120
3	0.76 (1.56)	0.40 (4.73)	0.98 (1.36)	0.21 (4.96)	0.75 (1.54)	1.331
4	0.84 (1.86)	0.51 (6.06)	1.08 (1.50)	0.24 (6.58)	0.82 (1.83)	1.407
5	0.93 (2.28)	0.63 (7.46)	1.20 (1.67)	0.27 (8.55)	0.91 (2.25)	1.635

Note: Numbers in brackets designate central factors of safety.

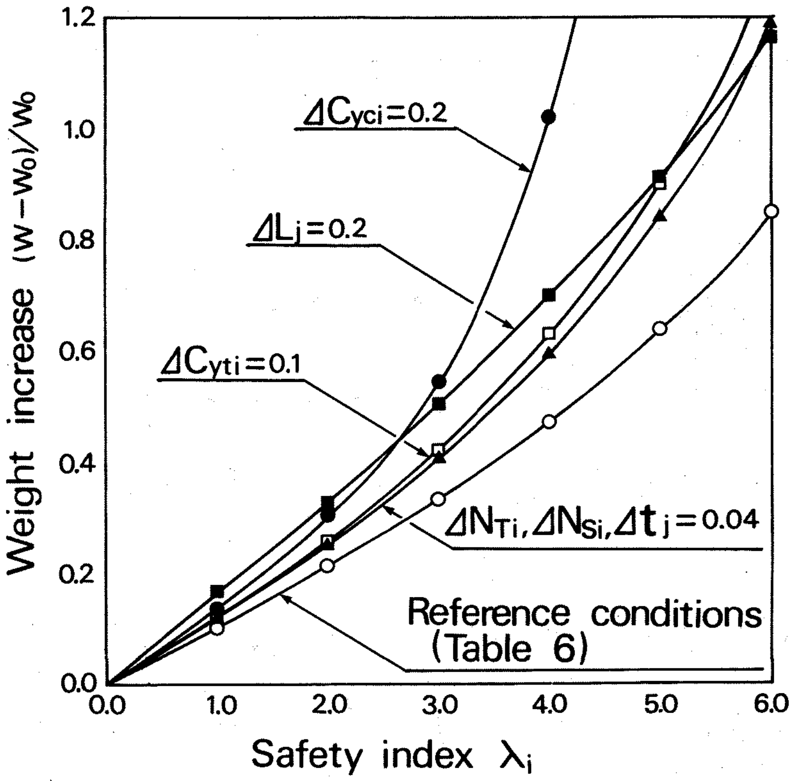

Fig. 3 Effect of variability on weight increase (Engine bed, W_0 : weight of optimum solutions at λ_i=0.0 of reference conditions in Table 6)

Example 3. One-bay two-story frame structure.

Plastic design of one-bay two-story frame structure is considered. Failure modes to be taken account of are shown in Fig. 4 and the following constraints are imposed on the relative strengths of the members:

$$\bar{R}_1 \le \bar{R}_6,\ \bar{R}_5 \le \bar{R}_6,\ \bar{R}_3 \le \bar{R}_1,\ \bar{R}_3 \le \bar{R}_5,\ \bar{R}_2 \le \bar{R}_3,\ \bar{R}_4 \le \bar{R}_3$$

The numbers added to the black circles of Fig. 4 correspond to those of collapse hinges. The safety margins are listed in Table 8. The design variables are the cross-sectional areas of the members. The moment capacities of the members are given[14] by

$$R_j = kCy_j A_j^{3/2}$$

where A_j, C_{yj} and k are the cross-sectional area, yield stress, of the j-th member and a constant, respectively. The weight is given by

$$W = \rho\{l_1(A_3+A_6)+l_3(A_1+A_5)+l_4(A_2+A_4)\}$$

where l_i are the lengths of the members.

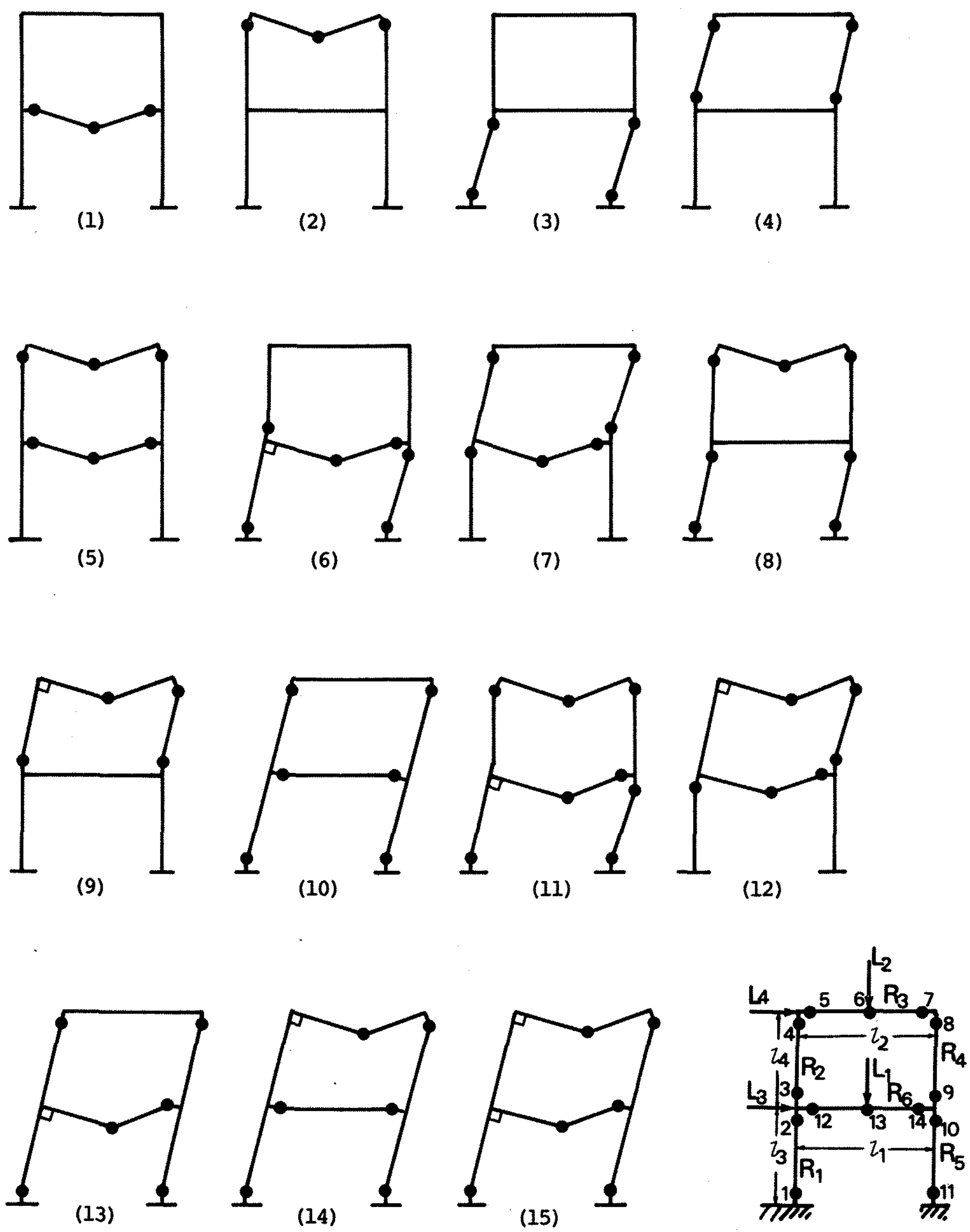

Fig. 4 One-bay two-story frame structure and failure modes.

Table 8 Safety margins of one-bay two-story frame structure

(a) Expressions for $\hat{T}_i$ and $\hat{S}_i$

Mode i	$\hat{T}_i$	$\hat{S}_i$
1	$M_{12}+2M_{13}+M_{14}$	$(l_1/2)L_1$
2	$M_4+2M_6+M_8$	$(l_2/2)L_2$
3	$M_1+M_2+M_{10}+M_{11}$	l_3L_3
4	$M_3+M_4+M_8+M_9$	l_4L_4
5	$M_4+2M_6+M_8+M_{12}+2M_{13}+M_{14}$	$(l_1/2)L_1+(l_2/2)L_2$
6	$M_1+M_3+M_{10}+M_{11}+2M_{13}+M_{14}$	$(l_1/2)L_1+l_3L_3$
7	$M_2+M_4+M_8+M_9+2M_{13}+M_{14}$	$(l_1/2)L_1+l_4L_4$
8	$M_1+M_2+M_4+2M_6+M_8+M_{10}+M_{11}$	$(l_2/2)L_2+l_3L_3$
9	$M_3+2M_6+M_8+M_9$	$(l_2/2)L_2+l_4L_4$
10	$M_1+M_4+M_8+M_{11}+M_{12}+M_{14}$	$l_3L_3+(l_3+l_4)L_4$
11	$M_1+M_3+M_4+2M_6+M_8+M_{10}+M_{11}+2M_{13}+M_{14}$	$(l_1/2)L_1+(l_2/2)L_2+l_3L_3$
12	$M_2+2M_6+M_8+M_9+2M_{13}+M_{14}$	$(l_1/2)L_1+(l_2/2)L_2+l_4L_4$
13	$M_1+M_4+M_8+M_{11}+2M_{13}+2M_{14}$	$(l_1/2)L_1+l_3L_3+(l_3+l_4)L_4$
14	$M_1+2M_6+M_8+M_{11}+M_{12}+M_{14}$	$(l_2/2)L_2+l_3L_3+(l_3+l_4)L_4$
15	$M_1+2M_6+M_8+M_{11}+2M_{13}+2M_{14}$	$(l_1/2)L_1+(l_2/2)L_2+l_3L_3+(l_3+l_4)L_4$

(b) Relations between plastic moments of hinges M_j and members R_j are given as follows:
$M_1=M_2=R_1$, $M_3=M_4=R_2$, $M_5=M_6=M_7=R_3$, $M_8=M_9=R_4$, $M_{10}=M_{11}=R_5$, $M_{12}=M_{13}=M_{14}=R_6$.

The data concerned are listed in Table 9. Corresponding to the various values of the safety indexes, the optimum designs are given in Table 10, which illustrates that the design variables move to the safety side as the safety indexes become large.

The effects of variability in allowable stresses, cross-sectional areas, loads and modeling errors on the optimum solutions are illustrated in Table 11, which shows that the values of the coefficients of variation influence the resulting optimum design.

Table 9 Data of one-bay two-story frame structure

(1) Length of members

l_1 mm	l_2 mm	l_3 mm	l_4 mm
6096	6096	4572	4572

(2) Coefficients of variation
in strength of materials...ΔC_{yj}=0.05
in fabrication errors......ΔA_j =0.02
in modelling errors........ΔN_{Ti}=0.10
........ΔN_{Si}=0.10

(3) Statistical data of loads

	1	2	3	4
$\bar{L}_j$	266.9 N	266.9 N	200.2 N	200.2 N
ΔL_j	0.1	0.1	0.1	0.1

(4) Data of materials
Yield stress .... $\bar{C}y_j$=2.482×10^8 Pa
Density .... ρ =7.833×10^3 kg/m^3

Table 10 Optimum solutions for various values of safety indexes
(One-bay two-story frame structure)

λ_i	$\bar{A}_1$ mm^2	$\bar{A}_2$ mm^2	$\bar{A}_3$ mm^2	$\bar{A}_4$ mm^2	$\bar{A}_5$ mm^2	$\bar{A}_6$ mm^2	W kg
0	8174.2	4729.0	8174.2	8174.2	8174.2	15458.0	181.4
1	8638.7	5548.4	8638.7	8638.7	8638.7	16109.6	192.4
2	9096.8	6387.1	9096.8	9096.8	9096.8	16735.5	203.3
3	9561.3	7270.9	9561.3	9561.3	9561.3	17341.9	214.4
4	10045.1	8219.3	10045.1	10045.1	10045.1	17954.8	225.9
5	10561.3	9258.0	10561.3	10561.3	10561.3	18574.2	238.2

Table 11 Effect of variability on optimum solutions (One-bay two-story frame structure, λ_i=4.0)

ΔC_{yj}	ΔL_j	ΔA_j	ΔN_{Ti}	ΔN_{Si}	$\bar{A}_1$ mm^2	$\bar{A}_2$ mm^2	$\bar{A}_3$ mm^2	$\bar{A}_4$ mm^2	$\bar{A}_5$ mm^2	$\bar{A}_6$ mm^2	W kg
0.05	0.10	0.02	0.10	0.10	10045.1	8219.3	10045.1	10045.1	10045.1	17954.8	225.93
0.10	0.10	0.02	0.10	0.10	10438.7	8851.6	10438.7	10438.7	10438.7	18419.3	234.68
0.10	0.20	0.02	0.10	0.10	10864.5	10793.5	10864.5	10864.5	10864.5	20735.4	255.25
0.05	0.10	0.02	0.20	0.10	12232.2	9774.2	12232.2	12232.2	12232.2	16290.3	252.18
0.05	0.10	0.02	0.10	0.20	10167.7	8554.8	10167.7	10167.7	10167.7	17754.8	227.69
0.05	0.10	0.04	0.10	0.10	10070.9	8264.5	10070.9	10070.9	10070.9	17987.1	226.51

6. Concluding Remarks

This paper is concerned with an optimum design of structural systems to minimize the structural weight based on safety index formats when various uncertainties in structural designs are taken into consideration. SLP (Sequential Linear Programming) is effectively applied to determining the optimum values of the design variables. Numerical examples are provided to demonstrate the validity of the proposed procedure.

However, for the optimum design procedure to be implemented in practice, the variability in various factors of the structural designs is to be consolidated and the values of the safety indexes must be determined by paying due attention to the safety and economy of the resulting structural system.

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Appendix 1. Interpretation of Safety Index

Introduce a standardized variate (Z_{si}) of the safety margin by the transformation:

$$Z_{si} = (Z_i - \bar{Z}_i)/\sigma_{Z_i} \tag{A.1}$$

By the use of Z_{si} , failure probability of the i-th failure mode is written as

$$\mathrm{Prob}[Z_i \leq 0] = \mathrm{Prob}[Z_{si} \leq -\bar{Z}_i/\sigma_{Z_i}] \tag{A.2}$$

Denote by $F_{Z_{si}}$ the probability distribution function of Z_{si}, *i.e.*,

$$F_{Z_{si}}(z) = \mathrm{Prob}[Z_{si} \leq z] \tag{A.3}$$

From (A.2) and (A.3), Eq.(9) is reduced to Eq.(7) and λ_i is related to P_{fai} as follows:

$$\lambda_i = -F^{-1}_{Z_{si}}(P_{fai}) \tag{A.4}$$

Appendix 2. Sequential Linear Programming

Sequential linear programming is a method of solving nonlinear programming problems, which uses a linear programming algorithm sequentially in such a way that in the limit the successive solutions of the linear programming problems converge to those of nonlinear programming problems. That is, in the successive stages, solve the linearized problem:

$$\text{Minimize:} \quad W(\bar{X}) - W(\bar{X}^{(k)}) = \sum_{j=1}^{n} \frac{\partial W}{\partial X_j} \delta X_j^{(k)}$$

with respect to $\delta X_j^{(k)}$ under the constraints:

$$\sum_{j=1}^{n} \left(\frac{\partial \bar{Z}_i}{\partial X_j} - \lambda_i \frac{\partial \sigma_{Z_i}}{\partial X_j}\right) \delta X_j^{(k)} \geq \lambda_i \sigma_{Z_i}(\bar{X}^{(k)}) - \bar{Z}_i(\bar{X}^{(k)})$$

$$(i=1,\ldots,m)$$

Adaptive move limits which limit the step size of $\delta X_j^{(k)}$ are used to secure the validity of the linear approximation and termination conditions on the successive changes in the design variables and the weight are also introduced to exclude the oscillation phenomena.